

Brane World Confronts Holography *

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Holography principle imposes a stringent constraint on the scale of quantum gravity M_* in brane-world scenarios, where all matter is confined on the brane. The thermodynamic entropy of astrophysical black holes and sub-horizon volumes during big bang nucleosynthesis exceed the relevant bounds unless $M_* > 10^{(4-6)}$ TeV, so a hierarchy relative to the weak scale is unavoidable. We discuss the implications for extra dimensions as well as holography.

I. INTRODUCTION

The idea that our universe might have extra dimensions beyond four space-time dimensions dates long back to early last century. In 1921 Kaluza [1] and later Klein [2] introduced extra dimensions to unify all forces in nature. The metric for higher dimensions was postulated to be

$$\hat{g}_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} g_{\mu\nu} - \varphi A_\mu A_\nu & \varphi A_\mu \\ \varphi A_\nu & \varphi \end{pmatrix}, \quad (1)$$

where $g_{\mu\nu}$ is the ordinary 4d metric and A_μ is interpreted as the photon field. After the dimensional reduction to $M^4 \times S^1$, the general coordinate transformation induces the $U(1)$ gauge transformation in M^4 .

String theory proposed as the theory of everything is consistent only in 10 dimensions, where 6 extra dimensions are compactified to be a Calabi-Yau manifold [3, 4].

Soon after Horava-Witten [5] pointed out that our world may be confined on a brane embedded in eleven dimensional spacetime, a low-scale gravity or a brane-world scenario [6, 7] is proposed as a solution to the gauge hierarchy problem. Letting gravity propagate in extra dimensions while all standard model particles are confined on a brane, the scale of gravity can be made arbitrary. In the brane world the Einstein action has the form

$$S = \int d^4x M_*^2 \mathcal{R} (d^{D-4}x M_*^{D-4}) \sqrt{-g} . \quad (2)$$

The relation between the fundamental Planck scale M_* and the apparent one M_P , the four-dimensional Planck

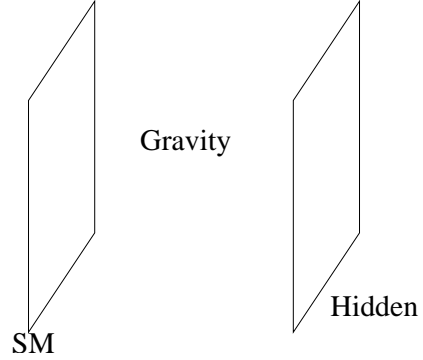


FIG. 1: Only gravity propagates in bulk.

scale, is given by

$$V_w M_*^{D-2} = M_P^2, \quad V_w \equiv \int d^{D-4}x \sqrt{-g_{(D-4)}}. \quad (3)$$

By adjusting V_w , the volume of the extra dimensions, we may take $M_* = O(1)$ TeV to solve the hierarchy problem. Since this low-scale gravity will be different from the usual gravity at short scales, Newton's law deviates at sub millimeter. The model is soon to be tested. However, in this talk we will argue that the holography bound requires the fundamental Planck scale can not be too small [8]. To reproduce the successful nucleosynthesis in Big Bang cosmology and to account for the supernova explosion, $M_* > 10^{4-6}$ TeV and the brane-world solution thus has a little hierarchy problem, even if it is operating.

II. WHAT IS HOLOGRAPHY BOUND

Bekenstein [9, 10] conjectured that for a system of energy M in a radius R , its entropy is bounded from above

$$S < \frac{2\pi M R}{\hbar}. \quad (4)$$

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For weak gravity, the size of the system is much larger than its Schwarzschild radius, $R_s (= 2GM) < R$. Therefore we get

$$S < \frac{2\pi MR}{\hbar} < \frac{A}{4G\hbar}, \quad A = 4\pi R^2. \quad (5)$$

The entropy of a system is less than one quarter of its area in the unit of Planck area, $l_p^2 = G\hbar/c^3$.

The Bekenstein bound for entropy lead 't Hooft [11] and Susskind [12] to formulate the holography principle, which states that the entropy in a spatial volume V enclosed by a surface area A cannot exceed $A/4$ in Planck units. Consider a system in a box. In quantum field theory, a state in a box can have arbitrarily large energy. However, if its compton length is smaller than its Schwarzschild radius, an observer outside the box can not access such a state. Therefore, the energy of states in the box is limited to outside observers. Such states do not contribute to entropy of the system measured by outside observers. For outside observers, the number of accessible states of the system is much less than that of the states allowed by a local quantum field theory.

In D dimensions, the Schwarzschild radius R_s of a system with energy E is determined roughly by the condition that the gravitational potential energy is of order one at $R = R_s$. If R_s is smaller than the radius of the extra dimensions,

$$\Phi \sim \frac{E}{M_*^{D-2} R^{D-3}} \longrightarrow R_s \sim (M_*^{2-D} E)^{1/(D-3)}. \quad (6)$$

Therefore, the energy of a system of size R must have an upper bound not to collapse into a black hole. If E_{\max} is the maximum energy of the system, then $E_{\max} < a^{-D} R^{D-1}$, where a^{-1} is the ultraviolet (UV) cutoff. Not to collapse into a black hole, the size of the system has to be bigger than its Schwarzschild radius

$$(M_*^{2-D} E_{\max})^{1/(D-3)} < (M_*^{2-D} a^{-D} R^{D-1})^{1/(D-3)} < R. \quad (7)$$

We find that the UV cutoff is related to the infrared cutoff of the system,

$$a > M_*^{-1} (R M_*)^{2/D}. \quad (8)$$

The entropy of the system [13] is given as ($k_B = 1$), neglecting all quantum numbers except positions,

$$S = \ln 2^{(R/a)^{D-1}} < (R M_*)^{D-3+2/D} \ln 2. \quad (9)$$

For $D = 4$, this bound gives $S < C A^{3/4}$, which is smaller than the area A .

III. COVARIANT ENTROPY BOUND

The Bekenstein bound for entropy applies to a static system only. For instance one can easily see that a closed universe or a collapsing star, where the system evolves in

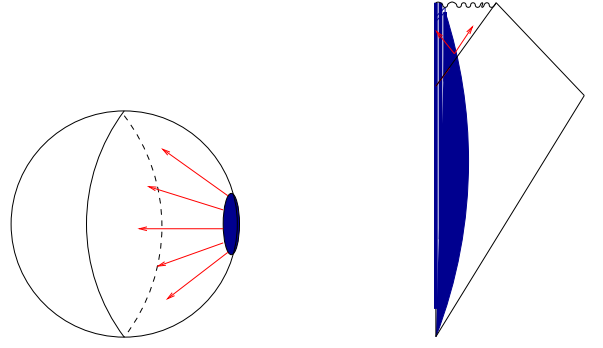


FIG. 2: Closed universe and a collapsing star

time, violates the Bekenstein bound (See Fig. 2). For a closed universe, near the big crunch, the area of the closed universe can be made arbitrarily small while its entropy never decreases. Similarly in the evolution of a collapsing star the spatial area of the collapsing star becomes arbitrarily small, violating the Bekenstein bound.

Since the spacelike volume does not have an intrinsic meaning in general relativity, one may introduce an intrinsic entropy bound [14]. In 1999 Bousso [15, 16] introduced a covariant entropy bound, which states that the entropy on any light-sheet of a surface B will not exceed the area of B :

$$S_L \leq \frac{A(B)}{4}. \quad (10)$$

The null geodesics extended from the surface will merge at a focal point in the future direction [17] (See Fig. 3), defining a null hypersurface L . As time evolves, all mat-

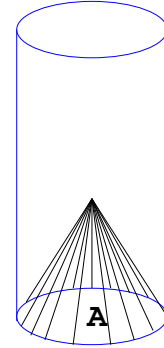


FIG. 3: Entropy on a null hypersurface, L .

ter inside B of spatial volume V will pass through the null hypersurface, L . By the second law of thermodynamics, the entropy in the null hypersurface is not less than the entropy inside B . Therefore, the covariant entropy bound gives the Bekenstein bound,

$$S_V < S_L < \frac{A}{4}. \quad (11)$$

Furthermore it does remain valid in the case of dynamical systems like collapsing universe or collapsing stars.

IV. BLACK HOLE THERMODYNAMICS

Original Bekenstein conjecture on entropy bound was motivated by black hole thermodynamics, where the entropy of black hole is found to be proportional to the area of the black hole horizon.

Here, we review the derivation of the black hole entropy by 't Hooft [18], which is based on two distinct properties of black holes. The first property is that black holes radiate as black bodies with a certain temperature, called Hawking temperature,

$$T_H = \frac{1}{8\pi M}, \quad (12)$$

where M is the mass of the black hole. The second property is that black holes have an event horizon. If one drops an object with energy ΔE into a black hole with mass E . ($\Delta E \ll 1 \ll E$ in Planck units.) Then, the absorption cross section is then

$$\sigma = \pi R^2, \quad R \simeq 2E. \quad (13)$$

From the Hawking's result, the emission probability is

$$W \simeq \pi R^2 \rho_{\Delta E} e^{-\beta_H \Delta E}, \quad (14)$$

where $\rho_{\Delta E}$ is the density of states for a particle with energy ΔE . Now, if we suppose the processes are described by a Hamiltonian acting in Hilbert space. Then,

$$\begin{aligned} \sigma &= |\langle E + \Delta E | T | E, \Delta E \rangle|^2 \rho(E + \Delta E) \\ W &= |\langle E, \Delta E | T | E + \Delta E \rangle|^2 \rho(E) \rho_{\Delta E}. \end{aligned} \quad (15)$$

By *PCT* invariance, the matrix elements have to be same and we get

$$\frac{\sigma}{W} = \frac{e^{\Delta E/T_H}}{\rho_{\Delta E}} = \frac{\rho(E + \Delta E)}{\rho(E) \rho_{\Delta E}} \quad (16)$$

Therefore, we find the density of states of the black hole $\rho(E) = \exp(4\pi E^2)$ and the black hole entropy becomes

$$S = \ln \rho(E) = 4\pi E^2 + S_0 \quad \text{or} \quad S = \frac{A}{4} + S_0. \quad (17)$$

where S_0 is the subleading term.

In the next section, we will apply the holography bound and the black hole entropy bound to the brane world scenarios, both ADD [6] and RS models [7].

V. HOLOGRAPHY BOUNDS ON BRANE WORLD SCENARIO

Consider an ADD/RS world in which the standard model degrees of freedom are confined to a 3-brane while the gravitational degrees of freedom propagate in D dimensions. The large effective volume V_w of the bulk allows the apparent Planck scale M_P to be much larger than the true dynamical scale of gravity $M_* \sim \text{TeV}$.

(1) *The holographic bound is violated during the big bang.*

Consider a spacelike region V of extent r_h on the 3-brane, and compare the apparent (3+1) entropy with the holographic bound applied to a hypersurface B which is the boundary of V . Let V have the same shape as the brane, with thickness of order M_*^{-1} , so that its surface area is of order r_h^2 in units of M_* . (It is possible that the brane is thicker than M_*^{-1} , forcing us to use a larger hypersurface with more entropy density, however it is hard to imagine that the brane thickness is parametrically larger than the fundamental length scale.) Impose that this region saturate the holographic bound, so r_h satisfies [19] (See Fig. 4)

$$T^3 r_h^3 \sim M_*^2 r_h^2, \quad (18)$$

or

$$r_h \sim T^{-1} \left(\frac{M_*}{T} \right)^2. \quad (19)$$

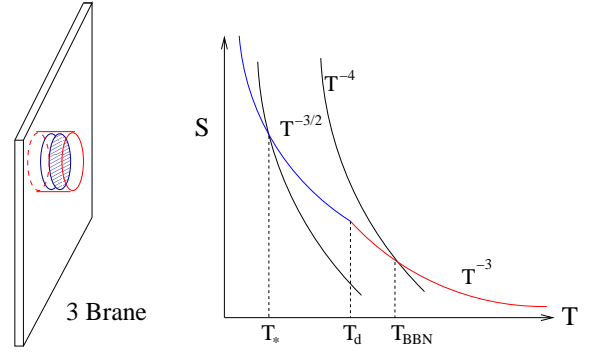


FIG. 4: Entropy in the horizon

Now consider a cosmological horizon volume of size $d_H \sim M_P/T^2$ (assuming radiation domination). The ratio of r_h to d_H is

$$\frac{d_H}{r_h} \sim \frac{T}{M_*} \frac{M_P}{M_*} \sim \frac{T}{10^{-4} \text{ eV}}. \quad (20)$$

For the matter-dominant epoch, the horizon distance is given as $d_H \sim (M_P/T_d^2) (T_d/T)^{3/2}$, where $T_d \simeq 10 \text{ eV}$ is the onset temperature of matter domination. The ratio then becomes

$$\frac{d_H}{r_h} \sim \frac{M_P}{M_*} \left(\frac{T^3}{M_*^2 T_d} \right)^{1/2} \sim \left(\frac{T}{10^{-2} \text{ eV}} \right)^{3/2}. \quad (21)$$

We find that for any temperature higher than 10^{-2} eV the causal horizon contains more degrees of freedom than are allowed according to the HB applied to the fundamental theory.

Our understanding of thermodynamics and statistical physics is based on counting states. If the HB is correct, the early universe in the brane worlds under consideration will likely not obey the usual laws of thermal

physics at temperatures $> 10^{-2}$ eV. This makes our understanding of nucleosynthesis and the microwave background problematic.

In order that our thermodynamic description of nucleosynthesis (at $T \sim 10$ MeV) not be invalidated by holography, we find that $M_* > 10^4$ TeV. (This bound is reduced slightly from (20) when prefactors in the expressions for the entropy density and horizon size are included.)

(2) *The holographic bound is violated by supernova cores.*

Consider the supernova of a star of mass $M > 8M_\odot$, which is powered by the collapse of an iron core and leads to neutron star or black hole formation. In this process the entropy of the collapsed neutron star is of order one per nucleon, so the total entropy is roughly 10^{57} . The radius of the core is a few to ten kilometers, so that its area (10^{12} cm²) in M_* units is only 10^{46} , where again we take a fiducial volume of thickness just greater than that of the brane. (As in the cosmological case the degrees of freedom we are counting are all confined to the brane.) Unless $M_* > 10^6$ TeV there is a conflict between the usual thermodynamic description of supernova collapse and the holographic entropy bound.

(3) *Black hole entropy bound vs. covariant bound*

Susskind [12] imagines a process in which a thermodynamic system is converted into a black hole by collapsing a spherical shell around it. Using the GSL, one obtains a bound on the entropy of the system: $S_{matter} \leq A/4$, where A is the area of the black hole formed. This is a *weaker* conjecture than the covariant bound, and has considerable theoretical support [9, 10, 12, 16]. In the application of the CB we are free to choose the hypersurface B, as long as its lightsheet intersects all of the matter whose entropy we wish to bound, whereas in the black hole bound the area which appears is that of the black hole which is formed. The black hole entropy bound is sensitive to the dynamics of horizon formation.

In TeV gravity scenarios, the black hole size on the 3-brane is controlled by the apparent Planck scale $M_P = 10^{19}$ GeV. The extent of the horizon in the perpendicular directions off the brane depends on the model, unless the hole is very small.

In ADD worlds, the horizon of an astrophysical black hole likely extends to the boundary of the compact extra dimensions. As discussed in [20], large black holes have geometry $S^2 \times T^{D-4}$, and the horizon includes all of the extra volume V_w . Due to this additional extra-dimensional volume, the resulting entropy density is the same as in 3+1 dimensions and there is no obvious violation of any bounds.

In RS scenarios, however, black holes are confined to the brane and have a pancake-like geometry [21, 22]. (See Fig. 5.) The black hole size in the direction transverse to the brane grows only logarithmically with the mass M . Thus far, no one has computed the Hawking temperature or entropy of a pancake black hole. In fact, exact solutions describing this objects have yet to be obtained. Let

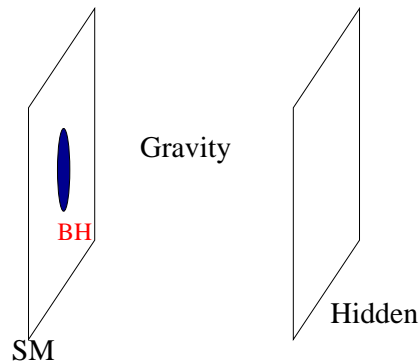


FIG. 5: Black holes in a brane world

us assume, motivated by holography, that the entropy of a pancake black hole continues to be of order its surface area in units of M_* . The surface area of a large hole is dominated by the $r^3 l^{D-5}$ component, so the black hole entropy bound arising from the Susskind construction in RS worlds is of the form

$$S < (rM_*)^3 . \quad (22)$$

That is, the upper bound on the entropy grows with the apparent 3-volume of the region. In this case the black hole bound is clearly weaker than the covariant bound, because the surface B used in the application of the latter is much smaller than the area of the pancake hole. Interestingly, (22) is the same result one would have obtained naively from $D = 3 + 1$ quantum field theory in the absence of gravity, with ultraviolet cutoff M_* !

VI. DISCUSSION

Our results can be interpreted in two ways, depending on how one views holography and related entropy bounds.

It seems likely that holography is a deep result of quantum gravity, relating geometry and information in a new way [16]. If so, it provides important constraints on extra dimensional models. Our analysis shows that the ordinary thermodynamic treatment of nucleosynthesis and supernovae are in conflict with the covariant bound. In other words, brane worlds obeying holography do not reproduce the observed big bang thermal evolution or stellar collapse. Exactly what replaces the usual behavior is unclear - presumably it is highly non-local - but the number of degrees of freedom is drastically less than in the thermodynamic description.

An alternative point of view is to regard brane worlds as a challenge to holography. If such worlds exist they have the potential to violate the entropic bounds by arbitrarily large factors. However, it must be noted that the basic dynamical assumptions underlying the scenarios (that the 3-brane and bulk geometry arise as a ground

state of quantum gravity) have never been justified. All violations discussed here require a hierarchy between M_P and M_* , or equivalently that the extra-dimensional volume factor $V_w = \int d^{D-4}x \sqrt{-g_{(D-4)}}$ exceed its “natural” size $\sim M_*^{-(D-4)}$.

Finally, we note that the brane, or whatever confines matter to 3 spatial dimensions, is absolutely necessary for these entropy violations. Without the brane, matter initially in a region with small extent in the extra $(D-4)$ dimensions will inevitably spread out due to the uncertainty principle. For ordinary matter in classical general relativity, in the absence of branes, Wald and collaborators [23] have proven the covariant entropy bound subject to some technical assumptions.

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